

GAUSSIAN QUEUES IN LIGHT AND HEAVY TRAFFIC

K. DEBICKI, K.M. KOSIŃSKI, AND M. MANDJES

ABSTRACT. In this paper we investigate Gaussian queues in the light-traffic and in the heavy-traffic regime. Let $Q_X^{(c)} \equiv \{Q_X^{(c)}(t) : t \geq 0\}$ denote a stationary buffer content process for a fluid queue fed by the centered Gaussian process $X \equiv \{X(t) : t \in \mathbb{R}\}$ with stationary increments, $X(0) = 0$, continuous sample paths and variance function $\sigma^2(\cdot)$. The system is drained with a constant rate $c > 0$, so that for any $t \geq 0$,

$$Q_X^{(c)}(t) = \sup_{-\infty < s \leq t} (X(t) - X(s) - c(t - s)).$$

We study $Q_X^{(c)} \equiv \{Q_X^{(c)}(t) : t \geq 0\}$ in the regimes $c \rightarrow 0$ (heavy traffic) and $c \rightarrow \infty$ (light traffic). We show for both limiting regimes that, under mild regularity conditions on σ , there exists a normalizing function $\delta(c)$ such that $Q_X^{(c)}(\delta(c)\cdot)/\sigma(\delta(c))$ converges to $Q_{B_H}^{(1)}(\cdot)$ in $C[0, \infty)$, where B_H is a fractional Brownian motion with suitably chosen Hurst parameter H .

1. INTRODUCTION

A substantial research effort has been devoted to the analysis of queues with Gaussian input, often also called *Gaussian queues* [10, 11, 12]. The interest in this model can be explained from the fact that the Gaussian input model is highly flexible in terms of incorporating a broad set of correlation structures and, at the same time, adequately approximates various real-life systems. A key result in this area is [18], where it is shown that large aggregates of Internet sources converge to a fractional Brownian motion (being a specific Gaussian process).

The setting considered in this paper is that of a centered Gaussian process $X \equiv \{X(t) : t \in \mathbb{R}\}$ with stationary increments, $X(0) = 0$, continuous sample paths and variance function $\sigma^2(\cdot)$, equipped with a deterministic, linear drift with rate $c > 0$, reflected at 0:

$$Q_X^{(c)}(t) = \sup_{-\infty < s \leq t} (X(t) - X(s) - c(t - s)).$$

The resulting *stationary workload process* can be regarded as a *queue* [14]. The objective of the paper is to study $Q_X^{(c)} \equiv \{Q_X^{(c)}(t) : t \geq 0\}$ in the limiting regimes $c \rightarrow 0$ (heavy traffic) and $c \rightarrow \infty$ (light traffic).

Date: January 18, 2013.

2010 Mathematics Subject Classification. Primary 60G15, 60F17; Secondary 60K25.

Key words and phrases. Gaussian processes, Heavy traffic, Light traffic, Functional limit theorems.

KD was supported by MNiSW Grant N N201 394137 (2009-2011) and by a travel grant from NWO (Mathematics Cluster STAR).

KK was supported by NWO grant 613.000.701.

Under mild conditions on the variance function $\sigma^2(\cdot)$, $Q_X^{(c)}$ is a properly defined, almost surely (a.s.) finite stochastic process. However, if $c \rightarrow 0$, then $Q_X^{(c)}(t)$ grows to infinity (in a distributional sense), for any $t \geq 0$. The branch of queueing theory investigating *how fast* $Q_X^{(c)}$ grows to infinity (as $c \rightarrow 0$) is commonly referred to as the domain of *heavy-traffic approximations*. In many situations this regime allows manageable expressions for performance metrics that are, under ‘normal’ load conditions, highly complex or even intractable, see for instance the seminal paper by Kingman [9] on the classical single-server queue. Since then, a similar approach has been followed in various other settings, see, e.g., [5, 13, 15, 17, 19] and many other papers.

Analogously, one can ask what happens in the *light-traffic* regime, i.e., $c \rightarrow \infty$; then evidently $Q_X^{(c)}$ decreases to zero. So far, hardly any attention has been paid to the light-traffic and heavy-traffic regimes for Gaussian queues. An exception is Dębicki and Mandjes [8], where the focus is on a special family of Gaussian processes, in a specific heavy-traffic setting. The primary contribution of the present paper concerns the analysis of $Q_X^{(c)}$ under both limiting regimes, for quite a broad class of Gaussian input processes X .

We now give a somewhat more detailed introduction to the material presented in this paper. It is well known that under the assumption that $\sigma(\cdot)$ varies regularly at infinity with parameter $\alpha \in (0, 1)$, for any function δ such that $\delta(c) \rightarrow \infty$ as $c \rightarrow 0$, there is convergence to fractional Brownian motion in the heavy-traffic regime:

$$(1) \quad \frac{X(\delta(c)\cdot)}{\sigma(\delta(c))} \xrightarrow{d} B_\alpha(\cdot), \text{ as } c \rightarrow 0.$$

We shall show that an analogous statement holds in the light-traffic regime, that is, if $\sigma(\cdot)$ varies regularly at zero with parameter $\lambda \in (0, 1)$ (i.e., $x \mapsto \sigma(1/x)$ varies regularly at infinity with parameter $-\lambda$), then for any function δ such that $\delta(c) \rightarrow 0$ as $c \rightarrow \infty$,

$$(2) \quad \frac{X(\delta(c)\cdot)}{\sigma(\delta(c))} \xrightarrow{d} B_\lambda(\cdot), \text{ as } c \rightarrow \infty.$$

Assuming that X satisfies some minor additional conditions, both (1) and (2) apply in $C(\mathbb{R})$, the space of all continuous functions on \mathbb{R} .

Our paper shows that the statements (1) and (2), which relate to the input processes, carry over to the corresponding stationary buffer content processes $Q_X^{(c)}$. That is, we identify, under specific conditions, a function $\delta(\cdot)$ such that

$$\frac{Q_X^{(c)}(\delta(c)\cdot)}{\sigma(\delta(c))} \xrightarrow{d} Q_{B_\alpha}^{(1)}(\cdot), \text{ as } c \rightarrow 0$$

and

$$\frac{Q_X^{(c)}(\delta(c)\cdot)}{\sigma(\delta(c))} \xrightarrow{d} Q_{B_\lambda}^{(1)}(\cdot), \text{ as } c \rightarrow \infty,$$

both in the space $C[0, \infty)$ of all continuous functions on $[0, \infty)$.

This paper is organized as follows. In Section 2 we introduce the notation and give some preliminaries. Subsection 3.1 presents the results for the heavy-traffic regime, whereas

Subsection 3.2 covers the light-traffic regime. We give the proofs of the main theorems, (i.e., Theorem 1 and Theorem 2) in Section 4.

2. PRELIMINARIES

In this paper we use the following notation. By $\text{id} : \mathbb{R} \rightarrow \mathbb{R}$ we shall denote the identity operator on \mathbb{R} , that is, $\text{id}(t) = t$ for every $t \in \mathbb{R}$. We write $f(x) \sim g(x)$ as $x \rightarrow x_0 \in [0, \infty]$ when $\lim_{x \rightarrow x_0} f(x)/g(x) = 1$. Let $\mathcal{RV}_\infty(\alpha)$ and $\mathcal{RV}_0(\lambda)$ denote the class of regularly varying functions at infinity with parameter α and at zero with parameter λ , respectively. That is, for a non-negative measurable functions f, g on $[0, \infty)$, $f \in \mathcal{RV}_\infty(\alpha)$ if for all $t > 0$, $f(tx)/f(x) \rightarrow t^\alpha$ as $x \rightarrow \infty$; $g \in \mathcal{RV}_0(\lambda)$ if for all $t > 0$, $g(tx)/g(x) \rightarrow t^\lambda$ as $x \rightarrow 0$.

2.1. Spaces of continuous functions. We refer to Billingsley [3] for the details of this subsection. For any $T > 0$, let $C[-T, T]$ be the space of all continuous functions $f : [-T, T] \rightarrow \mathbb{R}$. Equip $C[-T, T]$ with the topology of uniform convergence, i.e., the topology generated by the norm $\|f\|_{[-T, T]} := \sup_{t \in [-T, T]} |f(t)|$ under which $C[-T, T]$ is a separable Banach space. Therefore, by Prokhorov's theorem, weak convergence of random elements $\{X^{(c)}\}$ of $C[-T, T]$ as $c \rightarrow \infty$ is implied by convergence of finite-dimensional distributions and tightness. A family $\{X^{(c)}\}$ in $C[-T, T]$ is tight if and only if for each positive ε , there exists an a and c_0 such that

$$(3) \quad \mathbb{P}(|X^{(c)}(0)| \geq a) \leq \varepsilon, \text{ for all } c \geq c_0;$$

and, for any $\eta > 0$,

$$(4) \quad \lim_{\zeta \rightarrow 0} \limsup_{c \rightarrow \infty} \mathbb{P} \left(\sup_{\substack{|t-s| \leq \zeta \\ s, t \in [-T, T]}} |X^{(c)}(t) - X^{(c)}(s)| \geq \eta \right) = 0.$$

For notational convenience, we leave out the requirement $s, t \in [-T, T]$ explicitly in the remainder of this paper.

Finally, let $C(\mathbb{R})$ be the space of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f|_{[-T, T]} \in C[-T, T]$ for all $T > 0$. The above definitions extend in an obvious way to $C[0, T]$, $C[0, \infty)$ and convergence as $c \rightarrow 0$.

For $\gamma \geq 0$, let Ω^γ be the space of all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $\lim_{t \rightarrow \pm\infty} f(t)/(1 + |t|^\gamma) = 0$. Equip Ω^γ with the topology generated by the norm $\|f\|_{\Omega^\gamma} := \sup_{t \in \mathbb{R}} |f(t)|/(1 + |t|^\gamma)$ under which Ω^γ is a separable Banach space, so that Prokhorov's theorem applies. The following property can be found in [6, Lemma 3] or [7, Lemma 4].

Proposition 1. *Let a family of random elements $\{X^{(c)}\}$ on Ω^γ be given. Suppose that the image of $\{X^{(c)}\}$ under the projection mapping $p_T : \Omega^\gamma \rightarrow C[-T, T]$ is tight in $C[-T, T]$ for all $T > 0$. Then $\{X^{(c)}\}$ is tight in Ω^γ if and only if for any $\eta > 0$,*

$$(5) \quad \lim_{T \rightarrow \infty} \limsup_{c \rightarrow \infty} \mathbb{P} \left(\sup_{|t| \geq T} \frac{|X^{(c)}(t)|}{1 + |t|^\gamma} \geq \eta \right) = 0.$$

2.2. Fluid Queues. Let $Q_X^{(c)} \equiv \{Q_X^{(c)}(t) : t \geq 0\}$ denote a stationary buffer content process for a fluid queue fed by a centered Gaussian process $X \equiv \{X(t) : t \in \mathbb{R}\}$ with stationary increments, $X(0) = 0$, continuous sample paths and variance function $\sigma^2(\cdot)$. The system is drained with a constant rate $c > 0$, so that for any $t \geq 0$,

$$Q_X^{(c)}(t) = \sup_{-\infty < s \leq t} (X(t) - X(s) - c(t - s)).$$

Additionally, an equivalent representation for $Q_X^{(c)}(t)$ holds [16, p. 375]:

$$(6) \quad Q_X^{(c)}(t) = Q_X^{(c)}(0) + X(t) - ct + \max \left(0, \sup_{0 < s < t} \left(-Q_X^{(c)}(0) - (X(s) - cs) \right) \right).$$

Throughout the paper we say that X satisfies:

- C:** if $\sigma^2(t)|\log |t||^{1+\varepsilon}$ has a finite limit as $t \rightarrow 0$, for some $\varepsilon > 0$;
- RV₀:** if $\sigma \in \mathcal{RV}_0(\lambda)$, for $\lambda \in (0, 1)$;
- RV_∞:** if $\sigma \in \mathcal{RV}_\infty(\alpha)$, for $\alpha \in (0, 1)$;
- HT:** if both **C** and **RV_∞** are satisfied.
- LT:** if both **RV₀** and **RV_∞** are satisfied.

Remark 1. In our setting (X has stationary increments), the assumption that X is continuous is equivalent to the convergence of *Dudley integral*, see Subsection 2.3. This is immediately implied by condition **C**, see [1, Thm. 1.4]. However, the real importance of condition **C** lies in the fact that if in addition X satisfies **RV_∞**, then X also belongs to Ω^γ , for every $\gamma > \alpha$. This is pointed out in Subsection 3.1. Finally, note that **C** is met under **RV₀**. Indeed, since $\sigma \in \mathcal{RV}_0(\lambda)$, then $t \mapsto \sigma(1/t)$ belongs to $\mathcal{RV}_\infty(-\lambda)$, thus $\sigma^2(1/t)t^\lambda \rightarrow 0$ as $t \rightarrow \infty$. Equivalently, $\sigma^2(t)t^{-\lambda} \rightarrow 0$ as $t \rightarrow 0$, implying $\lim_{t \rightarrow 0} \sigma^2(t)|\log |t||^{1+\varepsilon} = 0$, for any fixed $\varepsilon > 0$. Furthermore, **RV_∞** implies that $X(t)/t \rightarrow 0$ a.s., for $t \rightarrow \pm\infty$, so that $Q_X^{(c)}$ is a properly defined stochastic process for any $c > 0$, see [7, Lemma 3]. Lastly, the assumption that X has continuous sample paths implies that σ is continuous.

Due to the stationarity of increments, all finite-dimensional distributions of X are specified by the variance function, since we have

$$(7) \quad \text{Cov}(X(t), X(s)) = \frac{1}{2} (\sigma^2(s) + \sigma^2(t) - \sigma^2(|t - s|)).$$

Recall that by $B_H \equiv \{B_H(t) : t \in \mathbb{R}\}$ we denote fractional Brownian motion with Hurst parameter $H \in (0, 1)$, that is, a centered Gaussian process with stationary increments, continuous sample paths, $B_H(0) = 0$ and covariance function

$$(8) \quad \text{Cov}(B_H(t), B_H(s)) = \frac{1}{2} (|s|^{2H} + |t|^{2H} - |t - s|^{2H}).$$

As mentioned in the introduction, if $c \rightarrow 0$, then, for any t , $Q_X^{(c)}(t) \rightarrow \infty$ a.s., which is called the *heavy-traffic regime*. On the other hand, if $c \rightarrow \infty$, then $Q_X^{(c)}(t) \rightarrow 0$ a.s., which is called the *light-traffic regime*.

2.3. Metric entropy. For any $\mathbb{T} \subset \mathbb{R}$ define the *semimetric*

$$d(t, s) := \sqrt{\mathbb{E}|X(t) - X(s)|^2} = \sigma(|t - s|), \quad t, s \in \mathbb{T}.$$

We say that $S \subset \mathbb{T}$ is a ϑ -net in \mathbb{T} with respect to the semimetric d , if for any $t \in \mathbb{T}$ there exists an $s \in S$ such that $d(t, s) \leq \vartheta$. The metric entropy $\mathbb{H}_d(\mathbb{T}, \vartheta)$ is defined as $\log \mathbb{N}_d(\mathbb{T}, \vartheta)$, where $\mathbb{N}_d(\mathbb{T}, \vartheta)$ denotes the minimal number of points in a ϑ -net in \mathbb{T} with respect to d . Later on we use the following proposition, see [2, Thm. 1.3.3] and [2, Corollary 1.3.4], respectively.

Proposition 2. *There exists a universal constant K such that for a d -compact set \mathbb{T}*

$$\mathbb{E} \left(\sup_{t \in \mathbb{T}} X(t) \right) \leq K \int_0^{\text{diam}(\mathbb{T})/2} \sqrt{\mathbb{H}_d(\mathbb{T}, \vartheta)} \, d\vartheta.$$

and for all $\zeta > 0$

$$\mathbb{E} \left(\sup_{\substack{(s,t) \in \mathbb{T} \times \mathbb{T} \\ d(s,t) < \zeta}} |X(t) - X(s)| \right) \leq K \int_0^\zeta \sqrt{\mathbb{H}_d(\mathbb{T}, \vartheta)} \, d\vartheta.$$

The quantity $\int_0^\infty \sqrt{\mathbb{H}_d(\mathbb{T}, \vartheta)} \, d\vartheta$ is called the Dudley integral.

3. MAIN RESULTS

In this section we formulate the result for the heavy-traffic and light-traffic regime, respectively. It is emphasized that these results are highly symmetric. Let us first introduce a function δ , such that for every $c > 0$

$$(9) \quad \frac{c\delta(c)}{\sigma(\delta(c))} = 1.$$

By the continuity of σ , we can choose δ as $\delta(c) = \inf\{x > 0 : x/\sigma(x) = 1/c\}$. From the definition of δ it follows that $\delta \in \mathcal{RV}_0(1/(\alpha - 1))$ under \mathbf{RV}_∞ and $\delta \in \mathcal{RV}_\infty(1/(\lambda - 1))$ under \mathbf{RV}_0 .

3.1. Heavy-traffic Regime. In the heavy-traffic regime we are interested in the analysis of $Q_X^{(c)}$ as $c \rightarrow 0$, under the assumption that X satisfies **HT**. The following statement follows from [7, Thms. 5 and 6].

Proposition 3. *If X satisfies **HT**, then*

$$\frac{X(\delta(c)\cdot)}{\sigma(\delta(c))} \xrightarrow{d} B_\alpha(\cdot), \text{ as } c \rightarrow 0,$$

in $C(\mathbb{R})$ and Ω^γ , for any $\gamma > \alpha$.

In fact, Proposition 3 holds for any function $\delta(c)$ such that $\delta(c) \rightarrow \infty$ as $c \rightarrow 0$. Condition **C** (which is one of the requirements of **HT**) plays a crucial role in proving tightness both in $C[-T, T]$, for some $T > 0$, and in Ω^γ .

Combining Proposition 3 with the definition of δ leads to the following statement.

Corollary 1. *If X satisfies **HT**, then*

$$\frac{X(\delta(c)\cdot) - c\delta(c)\text{id}(\cdot)}{\sigma(\delta(c))} \xrightarrow{d} B_\alpha(\cdot) - \text{id}(\cdot) \text{ as } c \rightarrow 0,$$

in $C(\mathbb{R})$.

Now we are in the position to present the main result of this subsection.

Theorem 1. *If X satisfies **HT**, then*

$$(10) \quad \frac{Q_X^{(c)}(\delta(c)\cdot)}{\sigma(\delta(c))} \xrightarrow{d} Q_{B_\alpha}^{(1)}(\cdot) \text{ as } c \rightarrow 0,$$

in $C[0, \infty)$.

We postpone the proof of Theorem 1 to Section 4.

Remark 2. Theorem 1 extends the findings of [8, Theorem 3.2] where, under the heavy-traffic regime, the weak convergence in $C[0, \infty)$ of $Q_X^{(c)}(\delta(c)\cdot)/\sigma(\delta(c))$ as $c \rightarrow 0$ was obtained for the class of input processes having differentiable sample paths a.s., i.e., of the form $X(t) = \int_0^t Z(s)ds$, where $\{Z(s) : s \geq 0\}$ is a stationary centered Gaussian process whose variance function satisfies specific regularity conditions.

3.2. Light-traffic Regime. In the light-traffic regime we analyze the convergence of $Q_X^{(c)}$ as $c \rightarrow \infty$, under the assumption that X satisfies **LT**. We begin by stating the counterpart of Proposition 3.

Proposition 4. *If X satisfies **RV**₀, then*

$$\frac{X(\delta(c)\cdot)}{\sigma(\delta(c))} \xrightarrow{d} B_\lambda(\cdot), \text{ as } c \rightarrow \infty,$$

*in $C(\mathbb{R})$. If, moreover, X satisfies **LT**, then the convergence also holds in Ω^γ , for any $\gamma > \max\{\lambda, \alpha\}$.*

Analogously to Proposition 3, Proposition 4 holds for any function $\delta(c)$ such that $\delta(c) \rightarrow 0$ as $c \rightarrow \infty$. As in the heavy-traffic case, combining Proposition 4 with the definition of δ leads to the counterpart of Corollary 1.

Corollary 2. *If X satisfies **RV**₀, then*

$$\frac{X(\delta(c)\cdot) - c\delta(c)\text{id}(\cdot)}{\sigma(\delta(c))} \xrightarrow{d} B_\lambda(\cdot) - \text{id}(\cdot) \text{ as } c \rightarrow \infty,$$

in $C(\mathbb{R})$.

The main result of this subsection is now stated as follows.

Theorem 2. *If X satisfies **LT**, then*

$$(11) \quad \frac{Q_X^{(c)}(\delta(c)\cdot)}{\sigma(\delta(c))} \xrightarrow{d} Q_{B_\lambda}^{(1)}(\cdot) \text{ as } c \rightarrow \infty,$$

in $C[0, \infty)$.

We postpone the proof of Proposition 4 and Theorem 2 to Section 4.

Remark 3. The assumption **LT** excludes the class of input processes of the structure $X(t) = \int_0^t Z(s)ds$, with $\{Z(s) : s \geq 0\}$ being a centered stationary Gaussian process with continuous sample paths a.s. (since $\lambda = 1$ in this case). In [8, Theorem 4.1] it was shown that, for this class of Gaussian processes, $Q_X^{(c)}(0)/\sigma(\delta(c))$ does *not* converge weakly to $Q_{B_\lambda}^{(1)}(0)$ as $c \rightarrow \infty$.

4. PROOFS

In this section we prove our results, but we start by presenting an auxiliary result.

Lemma 1. *If X satisfies **LT**, then for any $\epsilon > 0$, there exist constants $C, a > 0$, such that for all $x \leq a$ and $t > 0$,*

$$\frac{\sigma(tx)}{\sigma(x)} \leq C \times \begin{cases} t^\ell & t \leq 1, \\ t^u & t > 1, \end{cases}$$

where $\ell := \min\{\lambda - \epsilon, \alpha + \epsilon\}$ and $u := \max\{\alpha + \epsilon, \lambda + \epsilon\}$.

Proof Take any $\epsilon > 0$, then because $\sigma \in \mathcal{RV}_0(\lambda)$, there exists an $a \leq 1$ such that

$$(12) \quad \frac{\sigma(tx)}{\sigma(x)} \leq 2t^{\lambda-\epsilon}, \text{ for all } x \leq a \text{ and } tx \leq a.$$

Moreover, there exists a constant K_1 such that $\sigma(x) \geq K_1 x^{\lambda+\epsilon}$ for all $x \leq a$.

Because $\sigma \in \mathcal{RV}_\infty(\alpha)$, there exist constants $A, K_2 > 0$ such that $\sigma(x) \leq K_2 x^{\alpha+\epsilon}$ for all $x \geq A$. Because σ is continuous, we can in fact find a K_2 such that $\sigma(x) \leq K_2 x^{\alpha+\epsilon}$ for all $x \geq a$. Therefore

$$\frac{\sigma(tx)}{\sigma(x)} \leq \frac{K_2 (tx)^{\alpha+\epsilon}}{K_1 x^{\lambda+\epsilon}} =: K t^{\alpha+\epsilon} x^{\alpha-\lambda}, \text{ for all } x \leq a \text{ and } tx \geq a.$$

Note that, if $\alpha - \lambda \geq 0$, then we have

$$(13) \quad \frac{\sigma(tx)}{\sigma(x)} \leq K a^{\alpha+\epsilon} t^{\alpha+\epsilon}, \text{ for all } x \leq a \text{ and } tx \geq a.$$

If $\alpha - \lambda < 0$, then

$$(14) \quad \frac{\sigma(tx)}{\sigma(x)} \leq K a^{\alpha-\lambda} t^{\lambda+\epsilon}, \text{ for all } x \leq a \text{ and } tx \geq a.$$

Combining (12)–(14), we conclude that there exists a constant $C > 0$, such that

$$\frac{\sigma(tx)}{\sigma(x)} \leq C \max \left\{ t^{\lambda-\epsilon}, t^{\alpha+\epsilon}, t^{\lambda+\epsilon} \right\}, \text{ for all } x \leq a \text{ and all } t > 0.$$

□

In what follows, we will use the following notation. Let

$$X^{(c)}(t) := \frac{X(\delta(c)t)}{\sigma(\delta(c))}$$

and denote the variance of $X^{(c)}$ by $(\sigma^{(c)})^2$, that is,

$$\sigma^{(c)}(t) := \frac{\sigma(\delta(c)t)}{\sigma(\delta(c))}.$$

Proof of Proposition 4 We begin by showing the convergence in $C(\mathbb{R})$. To this end, we need to show the convergence in $C[-T, T]$ for any fixed $T > 0$.

Convergence in $C[-T, T]$: From the fact that $\sigma \in \mathcal{RV}_0(\lambda)$, it is immediate that the finite-dimensional distributions of $X^{(c)}$ converge in distribution to B_λ as $c \rightarrow \infty$, cf. (7)-(8), which also implies (3). Therefore, the weak convergence of $X^{(c)}$ in $C[-T, T]$ follows after showing (4).

By the Uniform Convergence Theorem, see [4, Thm. 1.5.2], for any $t \in (0, \zeta]$, we have $\sigma^{(c)}(t) \leq 2\zeta^\lambda$. Thus, Proposition 2 yields, for some universal constant $K > 0$,

$$\begin{aligned} \mathbb{P} \left(\sup_{|s-t| \leq \zeta} |X^{(c)}(t) - X^{(c)}(s)| \geq \eta \right) &\leq \mathbb{P} \left(\sup_{\sigma^{(c)}(|s-t|) \leq 2\zeta^\lambda} |X^{(c)}(t) - X^{(c)}(s)| \geq \eta \right) \\ &\leq \frac{1}{\eta} \mathbb{E} \left(\sup_{\sigma^{(c)}(|s-t|) \leq 2\zeta^\lambda} |X^{(c)}(t) - X^{(c)}(s)| \right) \\ &\leq \frac{K}{\eta} \int_0^{2\zeta^\lambda} \sqrt{\mathbb{H}^{(c)}([-T, T], \vartheta)} d\vartheta, \end{aligned}$$

where $\mathbb{H}^{(c)}([-T, T], \cdot)$ is the metric entropy induced by $\sigma^{(c)}$.

By Potter's bound [4, Thm. 1.5.6] for any $\epsilon, \zeta > 0$, $\epsilon < \lambda$ and $t \in (0, \zeta]$ and sufficiently large c (corresponding to small $\delta(c)$), we have $\sigma^{(c)}(t) \leq 2t^{\lambda-\epsilon}$. Hence

$$\mathbb{H}^{(c)}([-T, T], \vartheta) \leq \mathbb{H}_{\tilde{d}} \left([-T, T], \frac{\vartheta}{2} \right),$$

where \tilde{d} is a semimetric such that $\tilde{d}(s, t) = |t - s|^{\lambda-\epsilon}$. The inverse of $x \mapsto x^{\lambda-\epsilon}$ is given by $x \mapsto x^{1/(\lambda-\epsilon)}$, so that

$$\mathbb{H}_{\tilde{d}}([-T, T], \vartheta) \leq \log \left(\frac{T}{\vartheta^{1/(\lambda-\epsilon)}} + 1 \right) \leq C \log \left(\frac{1}{\vartheta} \right),$$

for some constant $C > 0$ and $\vartheta > 0$ small. It follows that

$$\int_0^{2\zeta^\lambda} \sqrt{\mathbb{H}^{(c)}([-T, T], \vartheta)} d\vartheta \leq \sqrt{C} \int_0^{2\zeta^\lambda} \sqrt{\log \left(\frac{2}{\vartheta} \right)} d\vartheta = 2\sqrt{C} \int_{\zeta^{-\lambda}}^\infty \frac{\sqrt{\log \vartheta}}{\vartheta^2} d\vartheta.$$

Summarizing, we have

$$\limsup_{c \rightarrow \infty} \mathbb{P} \left(\sup_{|s-t| \leq \zeta} |X^{(c)}(t) - X^{(c)}(s)| \geq \eta \right) \leq \frac{2K\sqrt{C}}{\eta} \int_{\zeta^{-\lambda}}^\infty \frac{\sqrt{\log \vartheta}}{\vartheta^2} d\vartheta;$$

we obtain (4) by letting $\zeta \rightarrow 0$.

Convergence in Ω^γ : To show the convergence in Ω^γ , we need to verify (5). Observe that

$$\begin{aligned} \mathbb{P} \left(\sup_{t \geq e^k} \frac{|X^{(c)}(t)|}{1+t^\gamma} \geq \eta \right) &\leq \frac{1}{\eta} \sum_{j=k}^{\infty} \frac{\mathbb{E} \sup_{t \in [e^j, e^{j+1}]} |X^{(c)}(t)|}{1+e^{j\gamma}} \\ &\leq \frac{1}{\eta} \sum_{j=k}^{\infty} \frac{\mathbb{E} |X^{(c)}(e^j)|}{1+e^{j\gamma}} + \frac{2}{\eta} \sum_{j=k}^{\infty} \frac{\mathbb{E} \sup_{t \in [e^j, e^{j+1}]} X^{(c)}(t)}{1+e^{j\gamma}} \\ &=: I_1(k) + I_2(k). \end{aligned}$$

$I_1(k)$ and $I_2(k)$ are dealt with separately. According to Lemma 1, for large c (that is, small $\delta(c)$), we have

$$\sigma^{(c)}(t) \leq C \times \begin{cases} t^\ell & t \leq 1, \\ t^u & t > 1, \end{cases}$$

where ℓ and u can be chosen such that $\ell, u < \gamma$. Therefore,

$$I_1(k) \leq \frac{1}{\eta} \sum_{j=k}^{\infty} \frac{\sigma^{(c)}(e^j)}{1+e^{j\gamma}} \leq \frac{C}{\eta} \sum_{j=k}^{\infty} \frac{e^{ju}}{1+e^{j\gamma}},$$

and the resulting upper bound tends to zero as $k \rightarrow \infty$.

Now focus on $I_2(k)$. For some universal constant $K > 0$ and because of the stationarity of the increments of X , Proposition 2 yields that $I_2(k)$ is majorized by

$$\frac{2K}{\eta} \sum_{j=k}^{\infty} \frac{\int_0^\infty \frac{\sqrt{\mathbb{H}^{(c)}([e^j, e^{j+1}], \vartheta)} \, d\vartheta}{1+e^{j\gamma}} = \frac{2K}{\eta} \sum_{j=k}^{\infty} \frac{\int_0^\infty \frac{\sqrt{\mathbb{H}^{(c)}([0, e^j(e-1)], \vartheta)} \, d\vartheta}{1+e^{j\gamma}}.$$

We will estimate the integrals under the sum by splitting the integration area into $\vartheta \leq 1$ and $\vartheta \geq 1$.

Observe that, for some constants $C_1, C_2 > 0$ (that is, not depending on j),

$$\begin{aligned} \int_0^1 \sqrt{\mathbb{H}^{(c)}([0, e^j(e-1)], \vartheta)} \, d\vartheta &\leq \int_0^1 \sqrt{\log \left(\frac{e^j(e-1)}{2\vartheta^{1/\ell}} + 1 \right)} \, d\vartheta \\ &\leq \int_0^1 \sqrt{C_1 + j + \frac{1}{\ell} \log \left(\frac{1}{\vartheta} \right)} \, d\vartheta \\ &= \ell e^{\ell(C_1+j)} \int_{C_1+j}^\infty \sqrt{\vartheta} e^{-\ell\vartheta} \, d\vartheta \\ &\leq \ell e^{\ell(C_1+j)} \int_0^\infty \sqrt{\vartheta} e^{-\ell\vartheta} \, d\vartheta = C_2 e^{\ell j}. \end{aligned}$$

Recall that $\ell < \gamma$, so that

$$\lim_{k \rightarrow \infty} \sum_{j=k}^{\infty} \frac{\int_0^1 \sqrt{\mathbb{H}^{(c)}([0, e^j(e-1)], \vartheta)} \, d\vartheta}{1+e^{j\gamma}} \leq \frac{2K}{\eta} \lim_{k \rightarrow \infty} \sum_{j=k}^{\infty} \frac{C_2 e^{\ell j}}{1+e^{j\gamma}} = 0.$$

So it remains to show the analogous statement for the integration interval $[1, \infty)$. Using a similar argumentation as the one above, one can show that

$$\int_1^\infty \sqrt{\mathbb{H}^{(c)}([0, e^j(e-1)], \vartheta)} d\vartheta \leq C_3 e^{uj},$$

for some constant $C_3 > 0$, from which the claim is readily obtained. \square

Since the proof of Theorem 1 is analogous to the proof of Theorem 2, we choose to focus on the light-traffic case only.

Proof of Theorem 2 The proof consists of three steps: convergence of the one-dimensional distributions, the finite-dimensional distributions, and a tightness argument.

Step 1: Convergence of one-dimensional distributions. In this step we show that, for a fixed $t \geq 0$,

$$\frac{Q_X^{(c)}(t)}{\sigma(\delta(c))} \xrightarrow{d} Q_{B_\lambda}^{(1)}(t), \text{ as } c \rightarrow \infty.$$

Since $Q_X^{(c)}$ is stationary, it is enough to show the above convergence for $t = 0$ only. Observe that, due to the time-reversibility property of Gaussian processes,

$$Q_X^{(c)}(0) \stackrel{d}{=} \sup_{t \geq 0} (X(t) - ct) = \sup_{t \geq 0} (X(\delta(c)t) - c\delta(c)t).$$

Upon combining Corollary 2 with the continuous mapping theorem, for each $T > 0$,

$$\sup_{t \in [0, T]} \left(\frac{X(\delta(c)t) - c\delta(c)t}{\sigma(\delta(c))} \right) \xrightarrow{d} \sup_{t \in [0, T]} (B_\lambda(t) - t), \text{ as } c \rightarrow \infty.$$

Thus it suffices to show that

$$(15) \quad \lim_{T \rightarrow \infty} \limsup_{c \rightarrow \infty} \mathbb{P} \left(\sup_{t \geq T} \left(\frac{X(\delta(c)t) - c\delta(c)t}{\sigma(\delta(c))} \right) \geq \eta \right) = 0,$$

for any $\eta > 0$. Recall the definition of $X^{(c)}$, so that

$$\mathbb{P} \left(\sup_{t \geq T} \left(\frac{X(\delta(c)t) - c\delta(c)t}{\sigma(\delta(c))} \right) \geq \eta \right) \leq \mathbb{P} \left(\sup_{t \geq T} \frac{|X^{(c)}(t)|}{\eta + t} \geq 1 \right),$$

where we used (9). Proposition 4 implies that the family $\{X^{(c)}\}$ is tight in Ω^γ , for some $\gamma \leq 1$. Now (15) follows from Proposition 1.

Step 2: Convergence of finite-dimensional distributions. The argumentation of this step is analogous to Step 1. First note that for any $t_i \geq 0$, $\eta_i > 0$ and $s_i < t_i$, where $i = 1, \dots, n$,

for any $n \in \mathbb{N}$, it follows that

$$\begin{aligned}
& \mathbb{P} \left(\frac{Q_X^{(c)}(\delta(c)t_i)}{\sigma(\delta(c))} > \eta_i, i = 1, \dots, n \right) \\
&= \mathbb{P} \left(\sup_{s \leq \delta(c)t_i} \left(\frac{X(\delta(c)t_i) - X(s) - c(\delta(c)t_i - s)}{\sigma(\delta(c))} \right) > \eta_i, i = 1, \dots, n \right) \\
&\leq \mathbb{P} \left(\sup_{s \in [s_i, t_i]} \left(\frac{X(\delta(c)t_i) - X(\delta(c)s) - c\delta(c)(t_i - s)}{\sigma(\delta(c))} \right) > \eta_i, i = 1, \dots, n \right) \\
&\quad + \sum_{i=1}^n \mathbb{P} \left(\sup_{s \leq s_i} \left(\frac{X(\delta(c)t_i) - X(\delta(c)s) - c\delta(c)(t_i - s)}{\sigma(\delta(c))} \right) > \eta_i \right).
\end{aligned}$$

Now the same procedure can be followed as in Step 1.

Step 3: Tightness in $C[0, T]$. In this step, for any $T > 0$, we show the tightness of $\{Q_X^{(c)}(\delta(c)\cdot)/\sigma(\delta(c))\}$ in $C[0, T]$. Given that we have established Step 2 already, (3) holds so we are left with proving (4), with $s, t \in [0, T]$; the remainder of the proof is devoted to settling this claim.

Stationarity of $Q_X^{(c)}$ implies that $\{Q_X^{(c)}(\delta(c)t) - Q_X^{(c)}(\delta(c)s) : t \geq s\}$ is distributed as

$$\{Q_X^{(c)}(\delta(c)(t - s)) - Q_X^{(c)}(0) : t \geq s\},$$

so that it suffices to prove (4) for $s = 0$ only. Furthermore, cf. (6),

$$\sup_{0 < t \leq \zeta} \left| Q_X^{(c)}(\delta(c)t) - Q_X^{(c)}(0) \right| \leq 2 \sup_{0 < t \leq \zeta} |X(\delta(c)t) - c\delta(c)t|.$$

From Corollary 2 it follows that

$$\sup_{0 < t \leq \zeta} \frac{|X(\delta(c)t) - c\delta(c)t|}{\sigma(\delta(c))} \xrightarrow{d} \sup_{0 < t \leq \zeta} |B_\lambda(t) - t|, \text{ as } c \rightarrow \infty.$$

Now notice that for $\zeta < \eta/4$, by the self-similarity of B_λ ,

$$\mathbb{P} \left(\sup_{0 < t \leq \zeta} |B_\lambda(t) - t| \geq \frac{\eta}{2} \right) \leq 2\mathbb{P} \left(\sup_{0 < t \leq 1} B_\lambda(t) \geq \frac{\eta}{4}\zeta^{-\lambda} \right).$$

Now it is straightforward to conclude that the last expression tends to zero as $\zeta \rightarrow 0$. \square

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INSTYTUT MATEMATYCZNY, UNIVERSITY OF WROCLAW, PL. GRUNWALDZKI 2/4, 50-384 WROCLAW, POLAND.

E-mail address: Krzysztof.Debicki@math.uni.wroc.pl

KORTEWEG-DE VRIES INSTITUTE FOR MATHEMATICS, UNIVERSITY OF AMSTERDAM, THE NETHERLANDS; EURANDOM, EINDHOVEN UNIVERSITY OF TECHNOLOGY

E-mail address: K.M.Kosinski@uva.nl

KORTEWEG-DE VRIES INSTITUTE FOR MATHEMATICS, UNIVERSITY OF AMSTERDAM, THE NETHERLANDS; EURANDOM, EINDHOVEN UNIVERSITY OF TECHNOLOGY, THE NETHERLANDS; CWI, AMSTERDAM, THE NETHERLANDS

E-mail address: M.R.H.Mandjes@uva.nl